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Renormalization of the Chapman–Enskog Expansion: Isothermal Fluid Flow and Rosenau Saturation

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This paper continues the author's study of procedures for rewriting the well-known Chapman–Enskog expansion used in the kinetic theory of gases. The usual Chapman–Enskog expansion, when used in isothermal fluid motion, will introduce nonlinear instability at super-Burnett order $O(\varepsilon^3)$ truncation. The procedure given here eliminates the truncation instability and produces the desired dissipation inequality.

KEY WORDS: Chapman–Enskog expansion; Burnett equation; Boltzmann equation; stability.

INTRODUCTION

The usual Navier–Stokes relation for the stress in one-dimensional motion of a monatomic gas is given by $\sigma = -p + \frac{4}{3}\mu u_x$ or in variational form $\sigma = -p + \partial_{(u_x/\rho)}\phi$

$$\phi = \frac{2\mu}{3}\int_{-\infty}^{\infty} \rho \left(\frac{u_x}{\rho}\right)^2 dx$$

where $\partial_{u_x/\rho}\phi$ denotes the Fréchet derivative with respect to u_x/ρ. The purpose of this paper is to show that an approximation (renormalization) to the complete sum of the Chapman–Enskog expansion for the stress of a gas of Maxwell molecules has a remarkably similar form, i.e., $\sigma_{rn} = -Lp + \partial_{(V_x/\rho)}\phi$ where

$$L = I - \frac{4}{3}\frac{\mu^2}{RT\rho}\frac{\partial}{\partial x}\frac{1}{\rho}\frac{\partial}{\partial x}, \qquad L\left(\frac{V_x}{\rho}\right) = \frac{u_x}{\rho}$$

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ϕ is a functional of ρ, V_x/ρ, and ρ_x/ρ^2, and $\partial_{V_x/\rho}\phi$ is its Fréchet derivative with respect to V_x/ρ. In addition the renormalized stress yields a natural "energy" dissipation relation and the viscous contribution to the stress exhibits for certain allowable parameter values a *finite* high frequency (large gradient) limit which I call *Rosenau saturation* [R1, R2, KR]. The importance of Rosenau saturation is that it provides a bounded viscous dissipation and is a closer approximation to the true Boltzmann equation in regions of large fluid gradients.

This work continues my investigation into the Chapman–Enskog expansion for the Boltzmann equation [Ce, Ch, G, TM, W]. The importance of the topic is well-known: the Chapman–Enskog expansion for the stress and heat flux serves as a formal bridge between the kinetic and continuum level descriptions of fluid flow. It provides a mechanism for describing the transition between the two levels which is crucial for flows that need to be described on a large range of macroscopic length and time scales.

It was first noted by Bobylev [B1] for Maxwell molecules and Luk'shin [L2–L3] for hard spheres and later in [F1, F2, Z] that the Chapman–Enskog approach fails when the expansion is truncated beyond some critical order: the truncated expansions yield the rest state of a monatomic gas governed by the Burnett $O(\varepsilon^2)$ and super-Burnett $O(\varepsilon^3)$ truncations linear unstable. (Here $\varepsilon > 0$ is the Knudsen number which play the role of the expansion parameter.) The instability is not a failure of the Chapman–Enskog expansion, it is only a failure of truncation. It was this failure in the truncations for the linearized theory that led Rosenau [R1] to give both a fundamental explanation of the instability phenomenon and suggest a method for rewriting the truncations and summing the entire Chapman–Enskog expansion for the Boltzmann equation. (A similar idea to Rosenau's was presented by Luk'shin [L1–L3] but Luk'shin did not attempt to sum the series to obtain an everywhere valid representation for the stress and heat flux.)

Gorban and Karlin [GK1] apparently unaware of Rosenau's paper suggested a summation technique similar to Rosenau's for the linearized Chapman–Enskog expansion. In [GK2] Gorban and Karlin continued their investigation and presented a framework for dealing with the completely nonlinear theory. That approach leads to similar but not identical results to the direct method given here and in [S]. For example, Gorban and Karlin's renormalized stress is not everywhere defined: it blows up at certain distinguished values of the velocity gradient.

In my earlier paper [S] I reconsidered Rosenau's program in a fully nonlinear context. In particular, I showed that the Chapman–Enskog expansion to super-Burnett order $O(\varepsilon^3)$ governing the temperature of a monatomic gas of Maxwellian molecules (i.e., r^{-5} attractive form between two molecules) is given by

$$\rho c_v T_t = \left(\varepsilon k_0 T T_x + \frac{\varepsilon^3 T^2}{16\rho_0^2} \left[813 \frac{T_x^3}{T^2} + 1451 \frac{T_x T_{xx}}{T} + 157 T_{xxx} \right] \right)_x + O(\varepsilon^4) \tag{0.1}$$

Here $T = T(x, t)$ is the variable temperature and a body force has been imposed to keep the density $\rho = \rho_0 > 0$ a constant and the velocity $u = 0$. In (0.1) the nonlinear $813(T_x^3/T^2)$ is stabilizing while the linear $157T_{xxx}$ is destabilizing. I showed in [S] that the destabilizing term $157T_{xxx}$ should be thought of as the second term in an asymptotic expansion for the resolvent of the operator

$$\frac{157}{16} \frac{\varepsilon^2 T^3}{\rho_0^2 k_0} \left(\frac{(\cdot)_x}{T^2} \right)_x$$

In particular applying the operator

$$M[F; T] = F - \frac{\lambda 157}{16} \frac{\varepsilon^2 T^3}{\rho_0^2 k_0} \left(\frac{F_x}{T^2} \right)_x$$

to both sides of (0.1) and retention of terms to $O(\varepsilon^3)$ was shown to yield the total generalized entropy

$$\int_{-\infty}^{\infty} \rho_0 \eta - \frac{\lambda}{2} \frac{157}{16} \frac{\varepsilon^2 c_v T^3}{k_0 \rho_0} \left(\frac{1}{T} \right)_x^2 dx$$

increasing when $\eta = c_v \log T$ for $\lambda_1 < \lambda < \lambda_2$, $\lambda_1 = 1.06447...$, $\lambda_2 = 16.581195...$. This showed the appropriate way to view the super-Burnett truncation for the heat conduction equation (0.1) is obtained after application of $M[F; T]$ to both sides of (0.1), and deletion of $O(\varepsilon^4)$ terms, i.e.,

$$\begin{aligned} \rho_0 c_v \left(T_t - \lambda \frac{157}{16} \frac{\varepsilon^2}{\rho_0^2 k_0} \left(T^3 \left(\frac{T_t}{T^2} \right)_x \right)_x \right) \\ = \left(\varepsilon k_0 T T_x + \frac{\varepsilon^3}{16\rho_0^2} \left[(813 + 2(157))\, T_x^3 + (1451 - 157)\, T T_x T_{xx} \right. \right. \\ \left. \left. + 157(1-\lambda)\, T^3 \left(\frac{(T T_x)_x}{T^2} \right)_x \right] \right)_x \end{aligned} \tag{0.2}$$

or in nonlocal form

$$\rho_0 c_v T_t + Q_x = 0 \tag{0.3}$$

with Q given by

$$Q(x,t) = -\int_{-\infty}^{\infty} G(x,\xi;T)\left(\varepsilon k_0 T T_x + \frac{\varepsilon^3}{16\rho_0^2}\left[(813+2(157))\,T_x^3 + (1451-157)\,T T_x T_{xx} + 157(1-\lambda)\,T^3\left(\frac{(T_x)_x}{T^2}\right)_x\right]\right) d\xi,$$

$$\lambda_1 < \lambda < \lambda_2. \tag{0.4}$$

Here G is the Green's function associated with the positive definite self-adjoint (on the weighted space $L^2(-\infty,\infty)$ with weight T^{-3}) $M[F]$.

In this paper I consider the problem complimentary to the heat conduction equation, i.e., I give a renormalization of the equations for isothermal fluid flow of a monatomic gas of Maxwellian molecules

$$\frac{D\rho}{Dt} + \rho u_x = 0 \tag{0.5}$$

$$\rho\,\frac{Du}{Dt} = (-p + \sigma_{vis})_x \tag{0.6}$$

which forces the fluid motion to do positive mechanical work. The temperature T is constant and the viscous stress is again given by the Chapman–Enskog expansion to super-Burnett order.

Remarkably the problem for the purely mechanical problem (0.5), (0.6) is also complimentary to the heat conduction problem in a surprising mathematical nature. In (0.1) the terms $813(T_x^3/T^2)$ and $157T_{xxx}$ were a nonlinear stabilizing term and a linear destabilizing term respectively. In (0.6) similar terms arise with T replaced by the velocity u in σ_{vis}. However, the roles are reversed: the nonlinear term is destabilizing and the linear term is stabilizing. Hence a related though different renormalization procedure must be invoked for the mechanical problem (0.5), (0.6).

As noted above I will show that the renormalization procedure leads to an exceptionally simple form for the renormalized isothermal stress:

$$\sigma_{rn} = -Lp + \partial_{(V_x/\rho)}\phi$$

where

$$L = I - \frac{4}{3}\frac{\mu^2}{RT\rho}\frac{\partial}{\partial x}\frac{1}{\rho}\frac{\partial}{\partial x}, \qquad L\left(\frac{V_x}{\rho}\right) = \frac{u_x}{\rho}$$

ϕ is a functional of ρ, V_x/ρ, and ρ_x/ρ^2, and $\partial_{V_x/\rho}\phi$ is its Fréchet derivative with respect to V_x/ρ. A reasonable conjecture is that this stabilizing, destabilizing coupling of linear and nonlinear terms will continue throughout the Chapman–Enskog expansion and methods given here and in [S] will produce stable truncations at every order and produce good approximations to the sum of the entire series for the viscous stress and heat conduction.

The paper has two sections after this one. In Section 1 I give an example which illustrates how a nonlinear viscous stress will yield truncation instability. The simple example captures the essence of the renormalization without extraneous technical difficulties. The example is similar to one given by Rosenau in [R2] as a model of a very weak viscous smoothing mechanism for interfaces. In Section 2 I consider the purely mechanical problem of one-dimensional isothermal steady fluid flow where the viscous stress is given by the Chapman–Enskog expansion for the Boltzmann equation with Maxwellian molecules. I show how the method given in Example 1, Section 1, will yield a renormalization of the viscous stress which does mechanical work.

1. AN ELEMENTARY EXAMPLE

An example will provide a simple way of visualizing instabilities in the Chapman–Enskog expansion.

Example 1. Consider the evolution equation

$$u_t = \varepsilon \left(\frac{u_x + \varepsilon^2 \beta u_x^3 - \alpha \varepsilon^2 u_{xxx}}{(1 + \varepsilon^2 u_x^2)^{1/2}} \right)_x, \qquad u \to 0 \quad \text{as } |x| \to \infty, \qquad \alpha, \beta > 0 \text{ constants.} \tag{1.1}$$

The rest state $u = 0$ is a solution. It is also a stable solution. For if we denote by (,) the $L^2(-\infty, \infty)$ inner product and $\|\cdot\|$ the associated $L^2(-\infty; \infty)$ norm we see

$$\frac{1}{2} \frac{d}{dt} \|u(t)\|^2 = -\varepsilon \left(u_x, \frac{u_x + \varepsilon^2 \beta u_x^3}{(1 + \varepsilon^2 u_x^2)^{1/2}} \right) - \alpha \varepsilon^3 \left\| \frac{u_{xx}}{(1 + \varepsilon^2 u_x^2)^{3/4}} \right\|^2 \leqslant 0$$

and so $\|u(t)\| \leqslant \|u(0)\|$. On the other hand $(1 + \varepsilon^2 u_x^2)^{-1/2}$ has the power series representation

$$(1 + \varepsilon^2 u_x^2)^{-1/2} = 1 - \frac{\varepsilon^2}{2} u_x^2 + \cdots \tag{1.2}$$

valid when $|\varepsilon u_x| < 1$. If we substitute the power series expansion into the evolution equation (1.1) we obtain formally

$$u_t = \left((\varepsilon u_x + \beta\varepsilon^3 u_x^3)\left(1 - \frac{\varepsilon^2}{2} u_x^2 + \cdots\right) - \alpha\varepsilon^3 u_{xxx}\right)_x + O(\varepsilon^4) \tag{1.3}$$

If we wish to study truncations of (1.3) we note for

$$O(\varepsilon)\colon u_t = (\varepsilon u_x)_x$$

which is the heat equation and $u = 0$ is again stable. However, for

$$O(\varepsilon^3)\colon u_t = (\varepsilon u_x + (\beta - \tfrac{1}{2})\, \varepsilon^3 u_x^3 - \alpha\varepsilon^3 u_{xxx})_x \tag{1.4}$$

Hence we see

$$\frac{1}{2}\frac{d}{dt} \|u(t)\|^2 = -\varepsilon \|u_x\|^2 + \varepsilon^3 \left(\frac{1}{2} - \beta\right) \|u_x^2\|^2 - \alpha\varepsilon^3 \|u_{xx}\|^2 \tag{1.5}$$

Here the first and third terms on the right side of (1.5) are stabilizing but the second is destabilizing for $\beta < \frac{1}{2}$ and stability of the rest state cannot be determined without additional information. (This is because the estimate

$$\|u_x^2\|^2 \leqslant \text{const. } (\|u_x\|^2 + \|u_{xx}\|^2)^2$$

on the destabilizing term cannot be compensated by the stabilizing terms without the imposition of small initial data for $|\varepsilon u_x|$.)

Hence the reason for the stability-instability paradox is clear: the restriction of smallness of $|\varepsilon u_x|$ must be enforced in (1.5) since that was a requirement for the convergence of the original geometric series.

Unfortunately in physical examples we do not have the luxury of knowing the original evolution equation to begin with. We can expect there is one, say similar to (1.1) which is *everywhere* valid producing a globally stable rest state, but we can only see its behavior via the dynamics of an asymptotic expansion similar to (1.3) which has a more *limited range of validity* and whose truncations seem to yield paradoxical stability information.

If we continue the expansion on the right-hand side of (1.4) we will obtain an equation of the form

$$u_t = q_x$$

where q has the form of a Chapman–Enskog asymptotic expansion

$$q = \sum_{n=0}^{\infty} \varepsilon^n q_n \tag{1.6}$$

and $q_n(u, ..., u^{(n)})$, $u^{(n)} = \partial_x u$, q_n is homogeneous of degree n:

$$q_n(u, \lambda u^{(1)}, \lambda^2 u^{(2)}, ..., \lambda^n u^{(n)}) = \lambda^n q_n(u, u^{(1)}, ..., u^{(n)})$$

for $\lambda \in C$.

Following the nomenclature of [S] the finite sum

$$\text{trunc}_N(q) = \sum_{n=0}^{N} \varepsilon^n q_n$$

is called the order N truncation of the Chapman–Enskog expansion. A renormalization of the Nth order truncation of the Chapman–Enskog expansion (1.6) is an asymptotic expansion

$$q^{rn} = \text{trunc}_N(q) + \sum_{n=N+1}^{\infty} \varepsilon^n \tilde{q}_n$$

where $\tilde{q}_n$ are homogeneous of degree n.

Finally, a renormalization is said to be $O((\varepsilon k)^p)$ as $k \to 0$, $k \to \infty$ if

$$\sum_{n=0}^{N} \varepsilon^n q_n(\bar{u}, e^{ikx}, ..., (ik)^n e^{ikx}) + \sum_{n=N+1}^{\infty} \varepsilon^n \tilde{q}_n(\bar{u}, e^{ikx}, ike^{ikx}, ..., (ik)^n e^{ikx}) = O((\varepsilon k)^p) \quad \text{as} \quad k \to 0,\ k \to \infty$$

for $\bar{u}$ constant. *In the case $p = 0$, $k \to \infty$ the renormalized expansion exhibits Rosenau saturation.* If a renormalized expansion is $O(\varepsilon k)$ as $k \to 0$, $k \to \infty$ then its response at high and low frequency imput is similar to the usual linear viscous relation $q = \varepsilon u_x$.

In the above example the original Chapman–Enskog expansion (1.4) can be seen to have the form

$$q = \varepsilon u_x + \varepsilon^3(\beta - \tfrac{1}{2})\, u_x^3 - \alpha\varepsilon^3 u_{xxx} + \cdots \tag{1.7}$$

and it does not lead to a stable rest state $u = 0$ at the $O(\varepsilon^3)$ truncation. However, it possesses an $N = 3$ renormalization expressed in (1.2)

$$q = (1 + \varepsilon^2 u_x^2)^{-1/2} (\varepsilon u_x + \varepsilon^3 \beta u_x^3 - \alpha\varepsilon^3 u_{xxx}) \tag{1.8}$$

which when written in asymptotic expansion as $\varepsilon \to 0$ agrees with the $O(\varepsilon^3)$ truncation (1.7) above. Furthermore, the $N=3$ renormalization (1.8) is $O(\varepsilon k)$ as $k \to 0$ and $O(\varepsilon^2 k^2)$ as $k \to \infty$.

A little inspection shows that the renormalization is not unique however. The choice

$$q = (1 + \varepsilon^2 \eta u_x^2)^{-\gamma} (\varepsilon u_x + \varepsilon^3 \beta u_x^2 - \alpha \varepsilon^3 u_{xxx})$$

will work just as well as long as $0 < \gamma \leqslant \frac{1}{2}$ and $\eta\gamma \geqslant \frac{1}{2}$. So our original choice $\gamma = \frac{1}{2}$, $\eta = 1$ is the one that leads to smallest choice of η and the smallest exponent p, $q = O((\varepsilon k)^p)$, as $k \to \infty$. That is the choice $\gamma = \frac{1}{2}$ is the one least effected by high frequency imput.

How should one then approach (1.3) to obtain the "hidden" evolution equation? Of course, the answer in this simple example is to sum the series (1.2) and then continue it analytically to all nonzero values of εu_x to recover (1.1) [CBV]. (A related analytic continuation idea was used by Bobylev in [B2].)

2. ISOTHERMAL MOTION OF A MONATOMIC GAS

In this section we will compute the renormalized viscous stress for a monatomic gas of Maxwellian molecules. First we note that the Chapman–Enskog expansion in one space dimension for the stress σ and heat flux q of a monatomic gas of Maxwellian molecules are given by the expansions (to super-Burnett order $N=3$ order [F1, F2, Fo, Z])

$$\begin{aligned}
\sigma = & -p + \frac{4}{3}\mu u_x - \frac{\mu^2}{p}\left[\frac{40}{27}u_x^2 - \frac{4}{3}\frac{RT}{\rho}\rho_{xx}\right.\\
& \left. + \frac{4}{3}\frac{RT}{\rho^2}\rho_x^2 - \frac{4}{3}\frac{R}{\rho}\rho_x T_x + 2\frac{R}{T}T_x^2 + \frac{2}{3}RT_{xx}\right]\\
& - \frac{\mu^3}{p^2}\left[\frac{47}{3}\frac{R}{\rho}T_x\rho_x u_x - \frac{64}{9}\frac{RT}{\rho^2}\rho_x^2 u_x + \frac{40}{9}\frac{RT}{\rho}\rho_{xx}u_x\right.\\
& - \frac{2}{3}\frac{RT}{\rho}\rho_x u_{xx} - \frac{21}{3}\frac{R}{T}T_x^2 u_x - \frac{47}{9}Ru_{xx}T_x\\
& \left. - \frac{31}{9}Ru_x T_{xx} + \frac{2}{9}RTu_{xxx} + \frac{16}{27}u_x^3\right] + O(\mu^4)
\end{aligned}$$

$$q = -\kappa T_x + \frac{\mu^2}{\rho}\left[\frac{95}{8T} u_x T_x - \frac{7}{4} u_{xx} - \frac{2}{\rho} u_x \rho_x\right]$$

$$+\frac{\mu^3}{p\rho}\left[-\frac{8035}{336}\frac{T_x}{T} u_x^2 + \frac{166}{21}\frac{\rho_x}{\rho} u_x^2 + \frac{989}{168} u_x u_{xx}\right.$$

$$+\frac{918}{8}\frac{R}{\rho T}\rho_x T_x^2 - \frac{1137}{16}\frac{R}{\rho^2} T_x \rho_x^2 + \frac{397}{16}\frac{R}{\rho}\rho_x T_{xx}$$

$$+\frac{701}{16}\frac{R}{\rho} T_x \rho_{xx} - \frac{813}{16}\frac{R}{T^2} T_x^3 - \frac{1451}{16}\frac{R}{T} T_x T_{xx}$$

$$\left.-\frac{157}{16} R T_{xxx} - \frac{41}{8}\frac{RT}{\rho^2}\rho_x \rho_{xx} - \frac{5}{8}\frac{RT}{\rho}\rho_{xxx} + \frac{23}{4}\frac{RT}{\rho^3}\rho_x^3\right] + O(\mu^4)$$

Here $\mu = \varepsilon T$, $\kappa = \kappa_0 \mu$ are the viscosity and coefficient of thermal conductivity respectively, ε is the Knudsen number, T is the temperature, ρ is the density, u is the velocity, p is the pressure $= R\rho T$, R is a gas constant, $\kappa_0 = \frac{3}{2} c_p$ where c_p is the specific heat at constant pressure and $c_p = \frac{5}{2} R$.

We now consider the usual balance laws of mass, momentum, and energy in one space dimension where a heat source is imposed to keep the flow isothermal (T = positive constant). In this case the relevant balance laws are

$$\frac{D\rho}{Dt} + \rho u_x = 0 \tag{2.1}$$

$$\rho \frac{Du}{Dt} = (-p + \sigma_{vis})_x \tag{2.2}$$

where from the Chapman–Enskog expansion we have

$$\sigma_{vis} = \frac{4}{3}\mu u_x - \frac{\mu^2}{p}\left[\frac{40}{27} u_x^2 - \frac{4}{3}\frac{RT}{\rho}\rho_{xx} + \frac{4}{3}\frac{RT}{\rho^2}\rho_x^2\right]$$

$$-\frac{\mu^3}{p^2}\left[-\frac{64}{9}\frac{RT}{\rho^2}\rho_x^2 u_x + \frac{40}{9}\frac{RT}{\rho}\rho_{xx} u_x - \frac{2}{3}\frac{RT}{\rho}\rho_x u_{xx}\right.$$

$$\left.+\frac{2}{9} RT u_{xxx} + \frac{16}{27} u_x^3\right] + O(\mu^4) \tag{2.3}$$

Notice the destabilizing $\frac{16}{27} u_x^3$ *term and stabilizing* $\frac{2}{9} RT u_{xxx}$ *appear in* (2.3) *just as in Example 1.* Before providing the renormalization procedure let us note the consequences of the use of the classical expansion.

If we use only the $O(\mu)$ term in σ_{vis} (2.1), (2.2) yields the compressible Navier–Stokes equations

$$\frac{D\rho}{Dt}+\rho u_x=0$$

$$\rho\frac{Du}{Dt}=\left(-p+\frac{4}{3}\mu u_x\right)_x$$

Multiplication of the second equation by u and integration from $-\infty$ to $+\infty$ in x yields the desired energy dissipation equation

$$\frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty}\rho(u^2+2RT\ln\rho)\,dx=-\int_{-\infty}^{\infty}\frac{4}{3}\mu u_x^2\,dx\leqslant 0$$

indicating the viscosity is dissipating mechanical energy.

Next consider the viscous stress σ_{vis} evaluated up to the $O(\mu^2)$ term. This yields the Burnett correction to the compressible Navier–Stokes equations

$$\frac{D\rho}{Dt}+\rho u_x=0,$$

$$\rho\frac{Du}{Dt}=\left(-p+\frac{4}{3}\mu u_x-\frac{\mu^2}{p}\left[\frac{40}{27}u_x^2-\frac{4}{3}\frac{RT}{\rho}\rho_{xx}+\frac{4}{3}\frac{RT}{\rho^2}\rho_x^2\right]\right)_x$$

A naive hope would be that multiplication of the second term in the above Burnett correction by u would also yield a dissipation relation similar to the one obtained for the compressible Navier–Stokes equation: velocity gradient terms in σ_{vis} should dissipate and the density dependent term $(-\frac{4}{3}(RT/\rho)\,\rho_{xx}+\frac{4}{3}(RT/\rho^2)\,\rho_x^2)_x$ which when multiplied by u should become the time rate of change of a capillarity energy term. Unfortunately, this is not the case: the $\frac{40}{27}u_x^2$ introduces an indeterminent form and the density dependent term does indeed produce a rate term, but as we shall see it is of a surprising non-local *velocity* form.

If we continue onto the $O(\mu^3)$ super-Burnett truncation of σ_{vis} we introduce the destabilizing $\frac{16}{27}u_x^3$ term and ability to obtain an analogue to the classical Navier–Stokes dissipation relation seems remote. Hence motivated by Example 1 we introduce the following renormalization procedure.

The first step in the renormalization procedure is to use the identity

$$\frac{1}{p^2}\left(\frac{2}{9}u_{xxx}RT-\frac{2}{3}\frac{u_{xx}\rho_x RT}{\rho}\right)$$
$$=\frac{2}{9}\frac{RT}{p^2}\rho^2\frac{\partial}{\partial x}\left(\frac{1}{\rho}\left(\frac{u_x}{\rho}\right)_x\right)+\frac{2}{9}\frac{u_x\rho_{xx}RT}{\rho p^2}-\frac{2}{3}\frac{u_x\rho_x^2 RT}{\rho^2 p^2}$$

to rewrite

$$\sigma_{vis} = \frac{4}{3}\mu u_x - \frac{\mu^2}{p}\left[\frac{40}{27}u_x^2 - \frac{4}{3}\frac{RT}{\rho}\rho_{xx} + \frac{4}{3}\frac{RT}{\rho^2}\rho_x^2\right]$$
$$-\frac{\mu^3}{p^2}\left[\frac{2}{9}RT\rho^2\frac{\partial}{\partial x}\left(\frac{1}{\rho}\left(\frac{u_x}{\rho}\right)_x\right)\right.$$
$$\left.+\frac{42}{9}\frac{u_x\rho_{xx}RT}{\rho} - \frac{70}{9}\frac{RT\rho_x^2u_x}{\rho^2} + \frac{16}{27}u_x^3\right] + O(\mu^4)$$

Next use the identity

$$\frac{u_x\rho_{xx}}{\rho^3} = \left(\frac{u_x\rho_x}{\rho^3}\right)_x + \frac{2u_x\rho_x^2}{\rho^4} - \frac{\rho_x}{\rho^2}\left(\frac{u_x}{\rho}\right)_x$$

to eliminate the $\frac{42}{9}(u_x\rho_{xx}RT/\rho)$ term in σ_{vis}:

$$\sigma_{vis} = \frac{4}{3}\mu u_x - \frac{\mu^2}{p}\left[\frac{40}{27}u_x^2 - \frac{4}{3}\frac{RT}{\rho}\rho_{xx} + \frac{4}{3}\frac{RT}{\rho^2}\rho_x^2\right]$$
$$-\mu^3\left[\frac{14}{9RT}\frac{u_x\rho_x^2}{\rho^4} + \frac{42}{9RT}\left(\frac{u_x\rho_x}{\rho^3}\right)_x - \frac{42}{9RT}\left(\frac{\rho_x}{\rho^2}\right)\left(\frac{u_x}{\rho}\right)_x\right.$$
$$\left.+\frac{2}{9RT}\left(\frac{1}{\rho}\left(\frac{u_x}{\rho}\right)_x\right)_x + \frac{16}{27}\frac{u_x^3}{p^2}\right] + O(\mu^4)$$

Now define the linear operator L:

$$L \doteq I - \frac{4}{3RT}\frac{\mu^2}{\rho}\frac{\partial}{\partial x}\frac{1}{\rho}\frac{\partial}{\partial x}$$

A simple computation shows

$$(L\varphi, \rho\psi) = (\rho\varphi, L\psi)$$

where (,) denotes the $L^2(-\infty, \infty)$ inner product, i.e. L is self-adjoint on the Sobolev space $H^2(-\infty, \infty)$ with respect to the $L^2(-\infty, \infty)$ inner product with weight ρ.

Since $p = R\rho T$, we note

$$Lp = p - \frac{4}{3}\frac{\mu^2}{p}\left(\frac{RT\rho_{xx}}{\rho} - \frac{RT\rho_x^2}{\rho^2}\right)$$

which is the crucial observation in renormalizing the μ^2 Burnett order term in σ_{vis}.

Hence we can write

$$\sigma = -Lp + \frac{4}{3}\mu u_x - \frac{\mu^2}{p}\frac{40}{27}u_x^2$$
$$-\mu^3\left[\frac{14}{9RT}\frac{u_x\rho_x^2}{\rho^4} + \frac{42}{9RT}\left(\frac{u_x\rho_x}{\rho^3}\right)_x - \frac{42}{9RT}\left(\frac{\rho_x}{\rho^2}\right)\left(\frac{u_x}{\rho}\right)_x\right.$$
$$\left. + \frac{2}{9RT}\left(\frac{1}{\rho}\left(\frac{u_x}{\rho}\right)_x\right)_x + \frac{16}{27}\frac{u_x^3}{p^2}\right] + O(\mu^4)$$

Now following the idea of Rosenau [R2] and the spirit of Example 1 we introduce the function

$$F\left(\rho, \frac{u_x}{\rho}, \frac{1}{\rho}\left(\frac{u_x}{\rho}\right)_x, \frac{\rho_x}{\rho^2}\right)$$
$$= 1 + \frac{\nu_1\mu^2}{(RT)^2}\left(\frac{u_x}{\rho}\right)^2 + \frac{\nu_2\mu^2}{(RT)}\left(\frac{\rho_x}{\rho^2}\right)^2 + \frac{\nu_3\mu^3}{(RT)^3}\left(\frac{u_x}{\rho}\right)^3 + \frac{\nu_4\mu^4}{(RT)^4}\left(\frac{u_x}{\rho}\right)^4$$
$$+ \frac{\nu_5\mu^4}{(RT)^3}\left(\frac{1}{\rho}\left(\frac{u_x}{\rho}\right)_x\right)^2 + \frac{\nu_6\mu^4}{(RT)^3}\frac{1}{\rho}\left(\frac{u_x}{\rho}\right)_x\left(\frac{u_x}{\rho}\right)\left(\frac{\rho_x}{\rho^2}\right)$$
$$+ \frac{\nu_7\mu^4}{(RT)^3}\left(\frac{u_x}{\rho}\right)^2\left(\frac{\rho_x}{\rho^2}\right)^2$$

From the definition of F we see

$$F^{-\gamma} = 1 - \frac{\gamma\nu_1\mu^2}{(RT)^2}\left(\frac{u_x}{\rho}\right)^2 - \frac{\gamma\nu_2\mu^2}{RT}\left(\frac{\rho_x}{\rho^2}\right)^2 + O(\mu^3)$$

where γ is an as yet undetermined constant. Hence

$$\frac{4}{3}\mu u_x F^{-\gamma} = \frac{4}{3}\mu u_x - \frac{4}{3}\gamma\nu_1\frac{\mu^3 u_x^3}{p^2} - \frac{4}{3}\mu^3\frac{\gamma\nu_2 u_x\rho_x^2}{RT\rho^4} + O(\mu^4)$$

and σ can be written as

$$\begin{aligned}\sigma = & -Lp + \frac{4}{3}\mu u_x F^{-\gamma} - \frac{\mu^2}{p}\frac{40}{27}u_x^2 F^{-\gamma} \\ & -\mu^3\left[\left(\frac{14}{9} - \frac{4\gamma v_2}{3}\right)\frac{u_x \rho_x^2 F^{-\gamma}}{RT\rho^4} + \frac{42}{9RT}\left(\frac{u_x \rho_x F^{-\gamma}}{\rho^3}\right)_x \right. \\ & - \frac{42}{9RT}\left(\frac{\rho_x}{\rho^2}\right)^2\left(\frac{u_x}{\rho}\right)_x F^{-\gamma} + \frac{2}{9RT}\left(\frac{1}{\rho}\left(\frac{u_x}{\rho}\right)_x F^{-\gamma}\right)_x \\ & \left. + \left(\frac{16}{27} - \frac{4\gamma v_1}{3}\right)\frac{u_x^3 F^{-\gamma}}{p^2}\right] + O(\mu^4)\end{aligned}$$

The next step is perhaps unmotivated at this juncture but will become clear shortly. Define

$$L^{-1}\left(\frac{u_x}{\rho}\right) \doteq \frac{V_x}{\rho}, \qquad \text{so that} \quad L\left(\frac{V_x}{\rho}\right) = \frac{u_x}{\rho}$$

(Since L is coercive from $H^2(-\infty, \infty)$ to $L^2(-\infty, \infty)$, L^{-1} exists as a bounded linear operator from $L^2(-\infty, \infty)$ onto $H^2(-\infty, \infty)$.) Then we see

$$\frac{u_x}{\rho} = \frac{V_x}{\rho} - \frac{4}{3RT}\frac{\mu^2}{\rho}\frac{\partial}{\partial x}\left(\frac{1}{\rho}\left(\frac{V_x}{\rho}\right)_x\right)$$

In particular if we assume $u \to 0$ as $|x| \to \infty$ we can integrate to find

$$u = V - \frac{4}{3RT}\frac{\mu^2}{\rho}\left(\frac{V_x}{\rho}\right)_x + \text{const}$$

and to force the constant to be zero we define V to be that solution of $L(V_x/\rho) = u_x/\rho$ which satisfies $V \to 0$ as $|x| \to \infty$.

We note σ can be expressed in terms of V_x instead of u_x:

$$\begin{aligned}\sigma = & -Lp + \frac{4}{3}\mu V_x \tilde{F}^{-\gamma} - \frac{\mu^2}{p}\frac{40}{27}V_x^2 \tilde{F}^{-\gamma} \\ & -\mu^3\left[\left(\frac{14}{9} - \frac{4\gamma v_2}{3}\right)\frac{V_x \rho_x^2 \tilde{F}^{-\gamma}}{RT\rho^4} + \frac{42}{9RT}\left(\frac{V_x \rho_x \tilde{F}^{-\gamma}}{\rho^3}\right)_x \right. \\ & - \frac{42}{9RT}\left(\frac{\rho_x}{\rho^2}\right)^2\left(\frac{V_x}{\rho}\right)_x \tilde{F}^{-\gamma} + \frac{2}{RT}\left(\frac{1}{\rho}\left(\frac{V_x}{\rho}\right)_x \tilde{F}^{-\gamma}\right)_x \\ & \left. + \left(\frac{16}{27} - \frac{4\gamma v_1}{3}\right)\frac{V_x^3}{p^2}\tilde{F}^{-\gamma}\right] + O(\mu^4)\end{aligned} \tag{2.4}$$

Notice now we have set $F(\rho, V_x/\rho, (1/\rho)(V_x/\rho)_x, \rho_x/\rho^2) = \tilde{F}$ and the term $(1/\rho)(V_x/\rho)_x \tilde{F}^{-\gamma}$ has the coefficient 2 arising from the substitution $u_x/\rho = L(V_x/\rho)$ in the $\frac{4}{3}\mu u_x F^{-\gamma}$ term.

The right-hand side of (2.4) can be expressed in a remarkably simple form. Consider the Fréchet derivative with respect to variation in (V_x/ρ) of the functional

$$\int_{-\infty}^{\infty} \frac{\rho(RT)^2 \tilde{F}^{1-\gamma}}{\mu} dx \doteq \phi\left(\rho, \frac{V_x}{\rho}, \frac{\rho_x}{\rho^2}\right)$$

Since the Euler–Lagrange equation associated with this functional ϕ is

$$\rho \frac{\partial \tilde{F}^{-\gamma}}{\partial (V_x/\rho)} - \frac{\partial}{\partial x}\left(\frac{\partial \tilde{F}^{1-\gamma}}{\partial (1/\rho)(V_x/\rho)_x}\right) = 0$$

we have the Fréchet derivative of ϕ with respect to the variation in V_x/ρ is

$$\partial_{(V_x/\rho)}\phi = \frac{\rho R^2 T^2}{\mu} \frac{\partial \tilde{F}^{1-\gamma}}{\partial (V_x/\rho)} - \frac{\partial}{\partial x}\left(\frac{R^2 T^2}{\mu} \frac{\partial \tilde{F}^{1-\gamma}}{\partial (1/\rho)(V_x/\rho)_x}\right)$$

We can compute $\partial_{(V_x/\rho)}\phi$ explicitly using the definition of $\tilde{F}$:

$$\frac{\partial \tilde{F}}{\partial (V_x/\rho)} = \frac{2\nu_1 \mu^2}{(RT)^2} \frac{V_x}{\rho} + \frac{3\nu_3 \mu^3}{(RT)^3}\left(\frac{V_x}{\rho}\right)^2 + 4\nu_4 \mu^4 \left(\frac{V_x}{\rho}\right)^3 + \frac{\nu_6 \mu^4}{RT} \frac{1}{\rho}\left(\frac{V_x}{\rho}\right)_x \left(\frac{\rho_x}{\rho^2}\right) + \frac{2\nu_7 \mu^4}{RT}\left(\frac{V_x}{\rho}\right)\left(\frac{\rho_x}{\rho^2}\right)^2$$

$$\frac{\partial \tilde{F}}{\partial (1/\rho)(V_x/\rho)_x} = \frac{2\nu_5 \mu^4}{RT} \frac{1}{\rho}\left(\frac{V_x}{\rho}\right)_x + \frac{\nu_6 \mu^4}{RT}\left(\frac{V_x}{\rho}\right)\left(\frac{\rho_x}{\rho^2}\right)$$

and

$$\partial_{(V_x/\rho)}\phi = \left(2\nu_1 \mu V_x + \frac{3\nu_3 \mu^2}{RT} \frac{V_x^2}{\rho} + \frac{4\nu_4 \mu^3}{(RT)^2} \frac{V_x^3}{\rho^2} + \frac{\nu_6 \mu^3}{RT}\left(\frac{V_x}{\rho}\right)_x \left(\frac{\rho_x}{\rho^2}\right) + \frac{2\nu_7 \mu^3 V_x}{RT}\left(\frac{\rho_x}{\rho^2}\right)^2\right)(1-\gamma)\tilde{F}^{-\gamma} - \frac{\partial}{\partial x}\left\{\left(\frac{2\nu_5 \mu^3}{RT} \frac{1}{\rho}\left(\frac{V_x}{\rho}\right)_x + \frac{\nu_6 \mu^3}{RT}\left(\frac{V_x}{\rho}\right)\frac{\rho_x}{\rho^2}\right)(1-\gamma)\tilde{F}^{-\gamma}\right\} \tag{2.5}$$

We now compare the right-hand sides of (2.4) and (2.5) and make the identifications

$$(1-\gamma)\,2\nu_1=\tfrac{4}{3},\qquad (1-\gamma)\,3\nu_3=-\tfrac{40}{27},\qquad (1-\gamma)\,4\nu_4=\tfrac{4}{3}\gamma\nu_1-\tfrac{16}{27},$$
$$(1-\gamma)\,\nu_6=\tfrac{42}{9},\qquad (1-\gamma)\,2\nu_7=\tfrac{4}{3}\gamma\nu_2-\tfrac{14}{9},\qquad (1-\gamma)\,2\nu_5=2$$

It then follows from that the isothermal stress has the simple representation

$$\sigma=-Lp+\partial_{(V_x/\rho)}\phi+O(\mu^4)$$

where

$$\begin{aligned}\partial_{(V_x/\rho)}\phi=&\frac{4}{3}\mu V_x\tilde{F}^{-\gamma}-\frac{\mu^2}{p}\frac{40}{27}V_x^2\tilde{F}^{-\gamma}-\mu^3\left(\frac{14}{9}-\frac{4\gamma\nu_2}{3}\right)\frac{V_x\rho_x^2\tilde{F}^{-\gamma}}{RT\rho^4}\\&+\frac{42}{9RT}\left(\frac{V_x\rho_x\tilde{F}^{-\gamma}}{\rho^3}\right)_x-\frac{42}{9RT}\left(\frac{\rho_x}{\rho^2}\right)\left(\frac{V_x}{\rho}\right)_x\tilde{F}^{-\gamma}\\&+\frac{2}{RT}\left(\frac{1}{\rho}\left(\frac{V_x}{\rho}\right)_x\tilde{F}^{-\gamma}\right)_x+\left(\frac{16}{27}-\frac{4\gamma\nu_1}{3}\right)\frac{V_x^3}{p^2}\tilde{F}^{-\gamma}\end{aligned}$$

Based on this computation we take the renormalized stress σ_{rn} to be

$$\sigma_{rn}=-Lp+\partial_{(V_x/\rho)}\phi \tag{2.6}$$

If we trace our steps backwards starting with σ_{rn} we see we recover the original Chapman–Enskog expansion to terms of order μ^4. Hence σ_{rn} is truly an $N=3$ renormalization of the Chapman–Enskog expansion of a gas of Maxwellian molecules for constant temperature.

We next will show that the evolution of the isothermal flow described by the renormalized stress σ_{rn} yields energy dissipation for certain choices of the as yet free parameters γ, ν_2. To do this we will need some simple integral identities.

First, we observe that since L is self-adjoint onto $H^2(-\infty,\infty)$ with respect to the $L^2(-\infty,\infty)$ inner product with weight ρ, then trivially L^{-1} is self-adjoint on $L^2(-\infty,\infty)$ with weight ρ. Hence

$$\begin{aligned}\left(u_x, L^{-1}\int_{-\infty}^{x}\rho\frac{Du}{Dt}dx\right)&=\left(\rho L^{-1}\left(\frac{u_x}{\rho}\right),\int_{-\infty}^{x}\rho\frac{Du}{Dt}dx\right)\\&=\left(\rho\frac{V_x}{\rho},\int_{-\infty}^{x}\rho\frac{Du}{DT}dx\right)=-\left(\rho V,\frac{Du}{Dt}\right)\end{aligned}\tag{2.7}$$

since $V\to 0$ as $|x|\to\infty$.

Next observe that

$$\frac{Du}{Dt}=\frac{DV}{Dt}-\frac{4}{3RT}\mu^2\left\{-\frac{1}{\rho^2}\frac{D\rho}{Dt}\left(\frac{V_x}{\rho}\right)_x+\frac{1}{\rho}\frac{D}{Dt}\left(\frac{V_x}{\rho}\right)_x\right\}$$
$$=\frac{DV}{Dt}-\frac{4}{3RT}\mu^2\left\{+\frac{1}{\rho}(\rho u_x)\left(\frac{V_x}{\rho}\right)_x+\frac{1}{\rho}\left(\frac{D}{Dt}\left(\frac{V_x}{\rho}\right)\right)_x-\frac{u_x}{\rho}\left(\frac{V_x}{\rho}\right)_x\right\}$$

and so

$$\frac{Du}{Dt}=\frac{DV}{Dt}-\frac{4\mu^2}{3RT\rho}\left(\frac{D}{Dt}\left(\frac{V_x}{\rho}\right)\right)_x \tag{2.8}$$

If we combine (2.7) and (2.8) we find

$$\left(u_x, L^{-1}\int_{-\infty}^{x}\rho\frac{Du}{Dt}dx\right)=-\left(\rho V,\frac{DV}{Dt}-\frac{4}{3RT}\frac{\mu^2}{\rho}\left(\frac{D}{Dt}\left(\frac{V_x}{\rho}\right)\right)_x\right)$$
$$=-\left(\rho V,\frac{DV}{Dt}\right)-\left(\rho\frac{4\mu^2}{3RT}\frac{V_x}{\rho},\frac{D}{Dt}\left(\frac{V_x}{\rho}\right)\right)$$

which by the classical transport relation

$$\frac{d}{dt}\int_{-\infty}^{\infty}\rho g\,dx=\int_{-\infty}^{\infty}\rho\frac{Dg}{Dt}dx$$

shows

$$\left(u_x, L^{-1}\int_{-\infty}^{x}\rho\frac{Du}{Dt}dx\right)=-\frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty}\rho\left(V^2+\frac{4}{3}\frac{\mu^2}{RT}\left(\frac{V_x}{\rho}\right)^2\right)dx \tag{2.9}$$

If we return to our balance laws of mass and momentum with stress given by σ_{rn} we see

$$\frac{D\rho}{Dt}+\rho u_x=0$$

$$\rho\frac{Du}{Dt}=(-Lp+\partial_{(V_x/\rho)}\phi)_x$$

and hence

$$\int_{-\infty}^{x}\rho\frac{Du}{Dt}dx=-Lp+\partial_{(V_x/\rho)}\phi$$

where we assume ρ and hence p go to zero as $|x| \to \infty$. Now we apply the linear operator L^{-1} to both sides of the above equation to obtain

$$L^{-1} \int_{-\infty}^{x} \rho \frac{Du}{Dt} dx = -p + L^{-1} \partial_{(V_x/\rho)} \phi$$

and hence taking the $L^2(-\infty, \infty)$ inner product of with u_x and using (2.9) we see

$$\frac{1}{2} \frac{d}{dt} \int_{-\infty}^{\infty} \rho \left(V^2 + \frac{4}{3} \frac{\mu^2}{RT} \left(\frac{V_x}{\rho} \right)^2 + 2RT \ln \rho \right) dx$$
$$= -(u_x, L^{-1} \partial_{(V_x/\rho)} \phi) \tag{2.10}$$

Note that is precisely at this point that the definition of V is crucial: since $(u_x, L^{-1}\partial_{(V_x/\rho)}\phi) = (\rho(u_x/\rho), L^{-1}\partial_{(V_x/\rho)}\phi) = (\rho L^{-1}(u_x/\rho), \partial_{(V_x/\rho)}\phi) = (V_x, \partial_{(V_x/\rho)}\phi)$ we have derived the energy identity

$$\frac{1}{2} \frac{d}{dt} \int_{-\infty}^{\infty} \rho \left(V^2 + \frac{4}{3} \frac{\mu^2}{RT} \left(\frac{V_x}{\rho} \right)^2 + 2RT \ln \rho \right) = -(V_x, \partial_{(V_x/\rho)} \phi) \tag{2.11}$$

Thus to obtain energy decay and when u, $\rho \to 0$ as $|x| \to \infty$ we need only find conditions to guarantee $(V_x, \partial_{(V_x/\rho)}\phi) \geq 0$.

If we compute $(V_x, \partial_{(V_x/\rho)}\phi)$ we find

$$(V_x, \partial_{(V_x/\rho)}\phi) = \int_{-\infty}^{\infty} \mu V_x^2 \tilde{F}^{-\gamma} \left\{ \frac{4}{3} - \frac{40}{27} \frac{V_x \mu}{p} - \left(\frac{16}{27} - \frac{4\gamma v_1}{3} \right) \frac{V_x^2 \mu^2}{p^2} \right\}$$
$$- \mu^3 \frac{\tilde{F}^{-\gamma}}{RT} \left\{ \left(\frac{14}{9} - \frac{4\gamma v_2}{3} \right) V_x^2 \left(\frac{\rho_x}{\rho^2} \right)^2 - \frac{42}{9} \frac{V_{xx}}{\rho} V_x \left(\frac{\rho_x}{\rho^2} \right) \right.$$
$$\left. - \frac{42}{9} V_x \left(\frac{\rho_x}{\rho^2} \right) \left(\frac{V_x}{\rho} \right)_x - 2 \frac{V_{xx}}{\rho} \left(\frac{V_x}{\rho} \right)_x \right\} dx \tag{2.12}$$

Now substitute

$$\frac{V_{xx}}{\rho} = \left(\frac{V_x}{\rho} \right)_x + \frac{V_x \rho_x}{\rho^2}$$

into (2.12) to obtain

$$(V_x, \partial_{(V_x/\rho)}\phi) = \int_{-\infty}^{\infty} \mu V_x^2 \tilde{F}^{-\gamma} \left\{ \frac{4}{3} - \frac{40}{27}\frac{V_x \mu}{p} - \left(\frac{16}{27} - \frac{4\gamma v_1}{3}\right) \frac{V_x^2 \mu^2}{p^2} \right\}$$
$$- \mu^3 \frac{\tilde{F}^{-\gamma}}{RT} \left\{ -\left(\frac{4\gamma v_2}{3} + \frac{28}{9}\right) V_x^2 \left(\frac{\rho_x}{\rho^2}\right)^2 \right.$$
$$\left. - \frac{102}{9} \left(\frac{V_x}{\rho}\right)_x V_x \left(\frac{\rho_x}{\rho^2}\right) - 2\left(\frac{V_x}{\rho}\right)_x^2 \right\} dx$$

Hence $(V_x, \partial_{(V_x/\rho)}\phi)$ is non-negative if the matrices A, B given by

$$A = \begin{bmatrix} \frac{4}{3} & -\frac{20}{27} \\ -\frac{20}{27} & \frac{4\gamma v_1}{3} - \frac{16}{27} \end{bmatrix}, \qquad B = \begin{bmatrix} \frac{2\gamma v_2}{3} + \frac{14}{9} & \frac{17}{6} \\ \frac{17}{6} & 1 \end{bmatrix}$$

and positive definite. Since $v_1 = 2/3(1-\gamma)$ the requirement $\det A > 0$ is equivalent to

$$1 > \gamma > \frac{61}{115} \doteq \gamma_{\text{crit}} \tag{2.13}$$

while the requirement $\det B > 0$ is equivalent to

$$v_2 > \frac{233}{24\gamma} \tag{2.14}$$

On the other hand F is a quadratic form and F and hence $\tilde{F}$ are positive definite when the matrices

$$C = \begin{bmatrix} v_1 & 0 & \frac{v_3}{2} \\ 0 & v_2 & 0 \\ \frac{v_3}{2} & 0 & v_4 \end{bmatrix}, \qquad D = \begin{bmatrix} v_5 & \frac{v_6}{2} \\ \frac{v_6}{2} & v_7 \end{bmatrix}$$

are positive definite. Substitution of the earlier obtained values for v_1, v_3, v_4, v_5, v_6, v_7 yields C and D positive definite when

$$\gamma > \frac{686}{1415} = \bar{\gamma} \tag{2.15}$$

and

$$\nu_2 > \frac{28}{3\gamma} \tag{2.16}$$

Since we need γ, ν_2 to satisfy (2.13)–(2.16) we require that γ, ν_2 satisfy (2.13), (2.14).

Hence when (2.14) and (2.15) are satisfied the "energy"

$$\frac{1}{2}\int_{-\infty}^{\infty} \rho\left(V^2 + \frac{4}{3}\frac{\mu^2}{RT}\left(\frac{V_x}{\rho}\right)^2 + 2RT\ln\rho\right)dx$$

is non-increasing in time. This is the generalization of the classical Navier–Stokes $O(\mu)$ energy dissipation.

This result is quite striking in that it delivers a critical exponent γ_{crit} for the expression for σ. It implies the renormalized stress has a singular diffusive component and the singularity must lie in the regime $(\gamma_{\text{crit}}, 1)$.

As the renormalization maps the non-local Boltzmann equation to a new non-local set of equations it unfortunately sheds no light on local boundary conditions for the new system.

Finally, as in Example 1 set $\rho_x/\rho^2 = ike^{ikx}$, $V_x/\rho = ike^{ikx}$, $(V_x/\rho)_x = (ik)^2 e^{ikx}$ and we see the renormalized viscous stress

$$\sigma_{vis.rn} \doteq \sigma_{rn} + L\rho$$

is $O(\varepsilon k)$ as $k \to 0$ and $O((\varepsilon k)^{3-4\gamma})$ as $k \to \infty$. Since $-1 < 3 - 4\gamma < 3 - 4\gamma_{\text{crit}} < \frac{101}{115}$ the renormalized viscous stress has sublinear growth in εk at high frequency imput. It also shows that the renormalization process is a very mild regularization procedure in that its effect at large k is even weaker than the classical $\sigma_{vis} = \frac{4}{3}\mu u_x$ relation. This suggests that renormalized equations possess a much weaker diffusion mechanism than classical compressible Navier–Stokes equations and is more realistic hydrodynamic model in that it will allow less penalization for the formation of inhomogeneities (shocks), again consistent with fundamental idea of Rosenau in [R2, KR]. Notice in the allowable case $\gamma = \frac{3}{4}$ the viscous stress exhibits Rosenau saturation and viscous stress has a finite high frequency limit.

In summary the isothermal motion governed by the balance laws of mass and momentum

$$\frac{D\rho}{Dt} + \rho u_x = 0,$$

$$\rho\,\frac{Du}{Dt} = (\sigma_{rn})_x$$

with $\sigma_{rn} = -Lp + \partial_{(V_x/\rho)}\phi$, $L(V_x/\rho) = u_x/\rho$ yields the energy

$$\frac{1}{2}\int_{-\infty}^{\infty} \rho\left(V^2 + \frac{4}{3}\frac{\mu^2}{RT}\left(\frac{V_x}{\rho}\right)^2 + 2RT\ln\rho\right)dx$$

non-increasing in time and $\sigma_{vis\ rn.} = \sigma_{rn} + Lp$ is $O(\varepsilon k)$ as $k \to 0$ and $O((\varepsilon k)^{3-4\gamma})$ as $k \to \infty$ where $1 > \gamma > \gamma_{\text{crit}} = 61/115$, $\nu_2 > 233/24\gamma$, $-1 < 3 - 4\gamma < 101/115$.

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